STAT/MATH 395 B **Homework 3**
Derby **Due August 19, 2009 at 8:30 am**

1. Use moment generating functions to solve the following problems:
   (a) *(4 points)* If $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$, what is the distribution of $Y = \sum_{i=1}^{n} X_i$?
   (b) *(4 points)* If $X \sim \text{N}(\mu, \sigma^2)$, what is the distribution of $Y = \left(\frac{X-\mu}{\sigma}\right)^2$?
   (c) *(4 points)* If $X_1, \ldots, X_n \overset{iid}{\sim} \text{N}(\mu, \sigma^2)$, what is the distribution of $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$?

2. On page 240 of the notes (Appendix C), Nate screwed up and listed the wrong moment generating function $M_X(t)$ for the Binomial distribution. Nate feels lucky that neither Lisa nor tita5289 have yet discovered this typo.
   (a) *(4 points)* The stated mgf is $(1-p)+pe^t$. What distribution is this the mgf for?
   (b) *(4 points)* Using your answer to part (a), what *should* the mgf of the binomial distribution be? Why?
   (c) *(4 points)* Now prove your answer to part (b) above by directly computing the mgf of the binomial distribution.
   (d) **Extra Credit:** *(1 point)* You probably used a combinatorial identity to solve part (c) above. What is the name of this combinatorial identity? (If you somehow (correctly) solved part (c) above without this identity, you automatically get this extra point)
   (e) **Extra Credit:** *(4 points)* Use this combinatorial identity to give another proof of 394 HW1 problem 12: In Example 1.1.6 on page 2 of the notes, it states that the number of possible subsets of a set S is 2^n, where $n = |S|$ =the size of the set S. Prove this.

3. *(6 points)* Let $X_1, X_2, \ldots, X_{12}$ be a random sample of size 12 from a standard Cauchy distribution (i.e., with $\theta = 0$).[1] What is the probability that exactly 5 of them will be positive (i.e., greater than zero)?

4. *(4 points)* Suppose that the joint pdf of X and Y is given by
$$f_{X,Y}(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \quad x \in (0,\infty),\ y \in (0,\infty).$$
   Compute $\text{E}[X|Y=y]$.

5. *(4 points)* Nate is trapped in his car in Los Angeles and wants to get out. He comes to an intersection and has three different directions he can go in. The first will lead him to leaving the city and getting on with his life. The second leads to a series of roads which will return him to that exact same spot after 5 hours of driving. The third leads to another series of roads will will return him to that exact same spot after 7 hours of driving. If we assume that Nate is at all times equally likely to choose any one of these directions (he's approaching 40, so his memory is getting really bad), what is the expected length of time until he finally leaves Los Angeles and gets on with his life?

[1] Note that there is also a typo for this distribution in Appendix C: There are no parameters α or β for this distribution.

6. Nate has decided to cancel the 395 final exam and instead issue each of his 32 students a random (continuous) score uniformly distributed between 0 and 110.

   (a) *(2 points)* What is the joint pdf of the minimal and maximal scores?

   (b) *(8 points)* What is the pdf of the range = maximal score − minimal score?

   (c) *(4 points)* What is the probability that the range will exceed 75?

7. An insurance company determines that their expected gain on a certain policy is \$12.

   (a) **Extra Credit:** *(2 points)* What is a lower estimate for the probability that their gain on this policy is under \$15?

   (b) **Extra Credit:** *(2 points)* Suppose we're also given that the standard deviation of their gain is \$3. What is a lower estimate for the probability that their gain on this policy is between \$4 and \$20?

8. Weyerhaeuser needs to know the probability p that a pine seedling planted in a cut area will survive for five years. They decide to conduct an experiment with some seedlings and use the proportion that survive for five years Y as an estimate for p. How many seedlings should they plant so that

$$\mathrm{P}(p - 0.01 < Y < p + 0.01) = 95\%?$$

   In other words, they want to be 95% certain that their experimental value is within 0.01 of the real p.

   (a) *(4 points)* Solve this using the weak law of large numbers. You may need to place a bound on the quantity pq.

   (b) *(4 points)* Solve this using the Central Limit Theorem. You may need to place a bound on the quantity pq.

   (c) *(2 points)* Which of the two answers above is smaller? Why?

9. **Extra Credit:** *(8 points)* Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Cauchy}(0)$. Show that no finite constant m exists to which the sample means $\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$ converge in probability. Does this disprove the law of large numbers?