STAT/MATH 395 B **Homework 2**
Derby **Due August 10, 2009 at 8:30 am**

1. Let $X_1 \sim \mathrm{U}(0, 4)$ be independent of $X_2 \sim \mathrm{U}(0, 2)$. EITHER find the pdf of the following functions using Jacobians OR tell me why it can't be done with Jacobians:

   (a) *(6 points)* $X_1 + X_2$.

   (b) *(6 points)* $X_1 - X_2$.

   (c) *(6 points)* $\min(X_1, X_2)$.

   (d) *(6 points)* $\max(X_1, X_2)$.

   (e) *(6 points)* X_1X_2.

2. Let $X_1 \sim \mathrm{U}(0, 4)$ be independent of $X_2 \sim \mathrm{U}(0, 2)$. Find the expected value and variance of the following random variables, by BOTH directly using the pdf found in problem 1 above (or problem 4 of HW1) AND by using a property of mathematical expectation along with the univariate pdfs of X_1 and X_2.

   (a) *(6 points)* $X_1 + X_2$.

   (b) *(6 points)* $X_1 - X_2$.

   (c) *(6 points)* X_1X_2.

   Determine the following:

   (d) *(4 points)* $\mathrm{E}[2X_1 - 9X_2 + 5]$.

   (e) *(4 points)* $\mathrm{E}[2X_1^2 - 9X_2^2 + 5]$.

   (f) *(4 points)* $\mathrm{Var}[2X_1 - 9X_2 + 5]$.

   (g) *(4 points)* $\mathrm{Cov}(2X_1, 9X_2)$.

3. A tire company promotes a protection plan insuring a new pair of its tires against blowouts for up to 50,000 miles. The cost is a one-time premium of $10.00 per pair of tires. The actual cost to the company for manufacturing, delivering, and installing one of these tires is $75.00. They know from experience that within 50,000 miles, the probability that a policy holder will have blowouts in one or both tires is 0.05 and 0.01, respectively.

   (a) *(3 points)* What is the company's expected gain or loss on each policy sold?

   (b) *(3 points)* What is the standard deviation of the gain or loss on each policy sold?

4. *(4 points)* Two white and two black dice are rolled, and the total is called X. Then the white dice are rolled again and their outcome is added to the total of the black dice (which were left unchanged from the first throw) – this total is called Y. What are the mean and variance of $Z = X + Y$?

5. Let $f_{X,Y}(x,y) = 12x^2, \quad 0 < x < y < 1$. Find the following:

   (a) *(6 points)* $\mathrm{E}[X]$.

   (b) *(6 points)* $\mathrm{E}[XY]$.

   (c) *(6 points)* $\mathrm{Cov}(X,Y)$.

6. *(6 points)* For a nonnegative integer-valued random variable X, show that

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \mathrm{P}(X \geq i).$$

7. **Extra Credit:** *(Google Interview Problem – 10 points)* Someone is going to feed you a sequence of objects (numbers, names, potential husbands, etc.) of unknown length, one at a time. Your job is to pick a random sample of exactly n of them, such that every number has an equal probability of being picked.

   To clarify, you get one number at a time, and you know how many you have to pick – but you have no idea how long the sequence will be. You don't know if the one you just got was the last one, or if there are a million more to come. At some point, you'll learn that there's no more input, and whatever you've chosen at that point is what you're stuck with. You need to be able to prove that each item in the sequence could have been picked with equal probability; it can't be that the first ones or last ones are even slightly more likely.

   The hard part is that you have no idea how long the sequence is – just that it will be at least n. You need to be able to prove that each item in the sequence could have been picked with equal probability - it can't be that the first ones or last ones are even slightly more likely.

   Clearly you're going to pick the first n items for your set. Then when the $(n+1)^{\text{st}}$ item arrives, with what probability should you pick it? And if you do, which item that you already have do you get rid of? Now extend this to the $(n+2)^{\text{nd}}$ item and so on.

8. **Extra Credit:** Prove the premises of Example 6.2.9 on page 101 of the notes. That is, if $X_1 = \sin(2\pi U)$ and $X_2 = \cos(2\pi U)$, where $U \sim \mathrm{U}(0,1)$, show that

   (a) *(4 points)* $\mathrm{Var}(X_1) = \frac{1}{2}$

   (b) *(4 points)* $\mathrm{Var}(X_2) = \frac{1}{2}$

   (c) *(4 points)* $\mathrm{Var}(X_1 + X_2) = 1$

   and then find

   (d) *(4 points)* $f_{X_1 X_2}(x_1, x_2)$

   (e) *(4 points)* $f_{X_1}(x_1)$

   (f) *(4 points)* $f_{X_2}(x_2)$

   so that we can properly conclude that even though $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$, it is nonetheless true that $f_{X_1 X_2}(x_1, x_2) \neq f_{X_1}(x_1) f_{X_2}(x_2)$.