STAT/MATH 395 B **Final Exam**
Derby **August 22, 2008**

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Name

1. Let $X \sim \text{Exp}(\lambda)$.

   (a) *(10 points)* Derive the moment generating function and the characteristic function for this distribution.

   (b) *(10 points)* Use either one to calculate $\text{E}[X]$ and $\text{Var}(X)$.

2. Let $X_1, X_2 \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$.

   (a) *(10 points)* Derive the pdf for $U = 4X_1 + 3X_2 - 3$. Show the derivation.

   (b) *(10 points)* Derive the pdf for $W = \left(\frac{X_1}{\sigma}\right)^2$. Show the derivation.

(c) *(10 points)* Derive the pdf for $Y = \left(\frac{X_1}{\sigma}\right)^2 + \left(\frac{X_2}{\sigma}\right)^2$. Show the derivation.

Note: If you aren't sure about the distribution of $\left(\frac{X_i}{\sigma}\right)^2$ from part (b), make an educated guess. Regardless of what you did in part (b), if you guess correctly on this and then proceed correctly, you will get full credit for this part.

3. Let $X_1, \ldots, X_{50} \overset{iid}{\sim} (\mu_X, \sigma_X^2)$, where $X_i \geq 0$, $X_i \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, and $\mu_X = 2$.

(a) *(10 points)* Obtain a lower bound on $\mathrm{P}\left(\sum_{i=1}^{50} X_i < 108\right)$ without making any approximations or assumptions about the distribution.

Now also assume that $\sigma_X^2 = \frac{1}{2}$.

(b) *(10 points)* Obtain a lower bound on $\mathrm{P}\left(92 < \sum_{i=1}^{50} X_i < 108\right)$ without making any approximations or assumptions about the distribution.

(c) *(5 points)* Use this new piece of information (i.e., about σ_X^2) to obtain a revised lower bound on $\mathrm{P}\left(\sum_{i=1}^{50} X_i < 108\right)$ without making any approximations or assumptions about the distribution. You should get a lower bound that is higher than the lower bound in part (a).

(d) *(10 points)* Approximate $\mathrm{P}\left(92 < \sum_{i=1}^{50} X_i < 108\right)$, and justify any assumptions you are making.

4. *(10 points)* It is the year 2018, and Nathan Powel and Eddie Heymann have designed an extremely fuel-efficient car that runs off of five independently-operated Energizer bunnies. This car is able to run as long as at least three of its five bunnies are functioning. If each bunny independently functions for a random amount of time that is distributed exponentially with a mean of 5 weeks, compute the pdf for the length of time that this car functions.

5. *(5 points)* Let $X \sim \text{Exp}(\lambda)$. Find the probability that X lies between the mean and the median of its distribution.

6. **Extra Credit:**

(a) *(2 points)* Why do you think the answers to Problems 3(b) and 3(d) are quite different?

(b) *(3 points)* Give three names (with their parameter values) for the distribution in Problem 2(c).

(c) *(1 point)* If a moment generating function $M_X(t)$ exists, it must be defined (i.e., finite) for all values of t within *a neighborhood around 0* – meaning, it must be valid for all $t \in (-\varepsilon, \varepsilon)$ for some value of $\varepsilon > 0$. Give a possible such value of ε for the moment generating function in Problem 1(a).

For the following, no proofs are needed for the following questions. If you don't know the answer, feel free to make an educated guess. You can give the pdf/pmf, or just give the name and parameters of the distribution.

(d) *(2 points)* If $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$, what is the distribution of $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$?

(e) *(2 points)* If $X_i \overset{ind}{\sim} \mathrm{N}(\mu_i, \sigma_i^2)$ for $i \in \{1, \ldots, n\}$ (where "$\overset{ind}{\sim}$" means "independently but *not* identically distributed as"), what is the distribution of $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$?