STAT/MATH 394 A | **Handout 1**
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Counting Problems on Different Levels

Frequently in counting problems we count in two or more different levels. Cards are an example of this, whereby we count the number of different *numbers* as well as the number of different *suits*. Example B.4.1 on page 235 of the notes is one complex example of this. Here we provide a few preliminary examples which lead up to that one.

1. Suppose we draw 2 cards from a normal 52-card deck. What is the probability that we get a pair – meaning, two cards of the same number, but different suits?

   Solution:

   First of all, the denominator of this fraction is $\binom{52}{2}$, since we are choosing 2 cards out of 52.

   For the numerator, we first pick a number out of the 13 available ones. For any number that we pick, we will choose 2 of the four suites:

   $$\underbrace{\binom{13}{1}}_{\substack{\text{choose}\\ \text{1 number}}} \cdot \underbrace{\binom{4}{2}}_{\substack{\text{choose}\\ \text{2 suits}}}$$

   Altogether, our probability is

   $$\frac{\binom{13}{1}\binom{4}{2}}{\binom{52}{2}}. \qquad \square$$

2. Suppose we draw 3 cards from a normal 52-card deck. What is the probability that we get a triple – meaning, three cards of the same number, but different suits?

   Solution:

   For similar logic as above, our denominator is $\binom{52}{3}$. For the numerator, we again pick a number out of the 13 available ones. For any number that we pick, we will choose 3 of the four suites:

   $$\underbrace{\binom{13}{1}}_{\substack{\text{choose}\\ \text{1 number}}} \cdot \underbrace{\binom{4}{3}}_{\substack{\text{choose}\\ \text{3 suits}}}$$

   Altogether, our probability is

   $$\frac{\binom{13}{1}\binom{4}{3}}{\binom{52}{3}}. \qquad \square$$

3. Suppose we draw 4 cards from a normal 52-card deck. What is the probability that we get two different pairs – meaning, for two different numbers?

Solution:

Now our denominator is $\binom{52}{4}$. For the numerator, we now pick two (different) numbers out of the 13 available ones. For any number that we pick for either of those, we will choose 2 of the four suites:

$$\underbrace{\binom{13}{2}}_{\substack{\text{choose}\\ \text{2 numbers}}} \cdot \underbrace{\binom{4}{2}}_{\substack{\text{choose}\\ \text{2 suits}}}^{2}$$

The $\binom{4}{2}$ term above is squared because we are doing this for each of the two numbers chosen. Altogether, our probability is

$$\frac{\binom{13}{2}\binom{4}{2}^2}{\binom{52}{4}}. \qquad \square$$

4. Suppose we draw 4 cards from a normal 52-card deck. What is the probability that we get a pair and two singletons – meaning, two of one number, then one of another number, and one of yet another number?

Solution:

Again,our denominator is $\binom{52}{4}$. For the numerator, we now pick three (different) numbers out of the 13 available ones. Of these three, we choose one to be for the pair – the other two will be for the singleton. We then pick two suits for the pair, and one for each of the two singletons:

$$\underbrace{\binom{13}{3}}_{\substack{\text{choose}\\ \text{3 numbers}}} \cdot \underbrace{\binom{3}{1}}_{\substack{\text{choose}\\ \text{1 number}\\ \text{for pair}}} \cdot \underbrace{\binom{4}{2}}_{\substack{\text{choose}\\ \text{2 suits}\\ \text{for pair}}} \cdot \underbrace{\binom{4}{1}}_{\substack{\text{choose}\\ \text{1 suit for}\\ \text{singletons}}}^{2}$$

The $\binom{4}{2}$ term above is squared because we are doing this for each of the two numbers chosen. Altogether, our probability is

$$\frac{\binom{13}{3}\binom{3}{1}\binom{4}{2}\binom{4}{1}^2}{\binom{52}{4}}. \qquad \square$$

5. Suppose we draw 5 cards from a normal 52-card deck. What is the probability that we get a pair and a triple?

Solution:

Now our denominator is $\binom{52}{5}$. For the numerator, we now pick two (different) numbers out of the 13 available ones. Of these two, we choose one to be for the pair – the other will be for the triple. Furthermore, as before, we pick 2 and 3 suits for the pair and triple, respectively:

$$\underbrace{\binom{13}{2}}_{\substack{\text{choose}\\ \text{2 numbers}}} \cdot \underbrace{\binom{2}{1}}_{\substack{\text{choose}\\ \text{1 number}\\ \text{for pair}}} \cdot \underbrace{\binom{4}{2}}_{\substack{\text{choose}\\ \text{2 suits}\\ \text{for pair}}} \cdot \underbrace{\binom{4}{3}}_{\substack{\text{choose}\\ \text{3 suits}\\ \text{for triple}}}$$

The $\binom{4}{2}$ term above is squared because we are doing this for each of the two numbers chosen. Altogether, our probability is

$$\frac{\binom{13}{2}\binom{2}{1}\binom{4}{2}\binom{4}{3}}{\binom{52}{5}}. \qquad \square$$

6. Suppose we draw 5 cards from a normal 52-card deck. What is the probability that we get a *flush* – meaning, that all five cards are of the same suit?

Solution:

Again, our denominator is $\binom{52}{5}$. For the numerator, we first pick one suit out of the four. Then we choose 5 numbers out of the 13:

$$\underbrace{\binom{4}{1}}_{\substack{\text{choose}\\ \text{1 suit}}} \cdot \underbrace{\binom{13}{5}}_{\substack{\text{choose}\\ \text{5 numbers}}}$$

Altogether, our probability is

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}. \qquad \square$$

7. Suppose we draw 5 cards from a normal 52-card deck. What is the probability that we get a *straight* – meaning, that all numbers are in a sequence, where the ace can be either high or low?

Solution:

Again, our denominator is $\binom{52}{5}$. For the numerator, we note that there are 10 sequences possible:

$$\begin{gathered} A, 2, 3, 4, 5 \\ 2, 3, 4, 5, 6 \\ 3, 4, 5, 6, 7 \\ \vdots \\ 9, 10, J, Q, K \\ 10, J, Q, K, A \end{gathered}$$

For each of these sequences, each of the 5 cards can be any of the 4 suits. Therefore, our numerator is simply

$$10 \cdot 4^5$$

so that our probability is

$$\frac{10 \cdot 4^5}{\binom{52}{5}}. \qquad \square$$