1. In this problem, we define "range" as "the set of all possible values."

   (a) *(2 points)* Let
   $$B = \{1, 2, 3, 4, 5, 6\}, \qquad C = B \times B$$
   and define the function
   $$g((x_1, x_2)) = x_1 + x_2, \qquad (x_1, x_2) \in C.$$
   What is the range of $g(\cdot)$?

   (b) *(2 points)* Let $A = \{0, 1\}$ and define $B = A \times A \times A$. For each element $(x_1, x_2, x_3) \in B$, let
   $$h((x_1, x_2, x_3)) = x_1 + x_2 + x_3.$$
   What is the range of $h(\cdot)$? How many elements have the value 0 assigned to them? How many have the value 1 assigned to them?

   (c) *(2 points)* Refer to the set U and the set function $n(A)$, defined as follows:
   $$U = \{1, 2, 3, 4, 5\}, \qquad n(A) = \text{ \# elements in } A\ .$$
   Let $\mathrm{P}(A) = \frac{n(A)}{5}\ \ \forall A \subset U$. What is the range of $\mathrm{P}(\cdot)$?

   (d) *(4 points)* Let U and $n(\cdot)$ be defined as above. Show that
   $$\begin{aligned} n(A) &\le n(B) && \text{if } A \subset B \\ n(A \cup B) &= n(A) + n(B) && \text{if } A \cap B = \varnothing. \end{aligned}$$
   The first of these says that the function $n(\cdot)$ is *monotonic*, while the second states that $n(\cdot)$ is *finitely additive.*

2. *(5 points)* A *computing stud* is someone who can program in SAS or LaTeX, and a *computing superstud* is someone who can program in both. Suppose we have a group of 47 computing studs, of which 36 program in SAS and 19 in LaTeX. How many computing superstuds are in this group?

3. *(10 points)* Let Ψ be the class of all subsets of the set of non-negative integers $\mathbb{Z}^*$. For $A \in \Psi$, define
   $$\mathrm{P}(A) = \sum_A p(x), \tag{1}$$
   where
   $$p(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x, \quad x \in \mathbb{Z}^*.$$
   Let
   $$\begin{aligned} &A_1 \equiv \mathbb{Z}^*, && A_2 \equiv \{4\}, && A_3 \equiv \{x:\ x = 4\}, && A_4 \equiv \{y:\ y = 4\}, \\ &A_5 \equiv \mathbb{Z}^*/\{0,1,2,3,4,5\}, && A_6 \equiv \mathbb{Z}^*/\{0,1,2,\ldots,n\}, && A_7 \equiv \{x:\ x = 2k,\ k \in \mathbb{Z}^*\}, && A_8 \equiv \{x:\ x = nk,\ k \in \mathbb{Z}^*\}, \end{aligned}$$
   where $n \in \mathbb{Z}^+$. Find $\mathrm{P}(A_i)$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. What is noteworthy about your answers that might have something to do with probability?

4. *(10 points)* For events A, B, C and D, prove that $(A^c B^c)^c = A \cup B$ and $(C^c \cup D^c)^c = CD$.

5. *Washington State License Plates.* From 1958 until about now, Washington State non-vanity license plates were of the form `XXX-YYY`, where `X` is any digit from 0 to 9 and `Y` is any (Roman) letter from A to Z.[1]

   (a) *(2 points)* Describe the sample space S of all possible license plate combinations.

   (b) *(2 points)* What is the size of S?

   (c) *(2 points)* In reality, there are 2417 objectionable three-letter combinations (even including such innocent words as `CAT`, `KID` and `ZOO`). With these restrictions, what is the size of the sample size S?

   (d) *(2 points)* Furthermore, letter combinations do not start with I, O or Q. Assuming that none of the 2417 objectionable three-letter combinations listed above start with these letters (which might be false: If `CAT` is objectionable, then `ONE` probably is), what is the size of the sample size S?

   Starting this summer, license plates are now being made with four letters. Letter combinations still do not start with I, O or Q. Assume that there are 1973 objectionable four-letter combinations.

   (e) *(2 points)* With this new configuration included in the sample space, what is the size of the new sample space S?

   (f) With this new sample space, what is the probability of getting the following license plate numbers:

      i. *(2 points)* Starting with `394`.
      ii. *(2 points)* Just even digits.
      iii. *(2 points)* Just even distinct digits.
      iv. *(3 points)* Only letters from the word `PROBABILITY` (assuming this doesn't result in any of the 2417 or 1973 objectionable words).
      v. *(3 points)* Only distinct letters from the word `PROBABILITY` (assuming this doesn't result in any of the 2417 or 1973 objectionable words).
      vi. **Extra Credit:** *(10 points)* Ending with four letters, whereby the last three are any letters from the word `PROBABILITY`.
      vii. **Extra Credit:** *(5 points)* Ending with four letters, whereby the last three are any distinct letters from the word `PROBABILITY`.
      viii. *(2 points)* Ending with `EVAN`.
      ix. *(2 points)* Ending with `OWEN`.

   (g) *(1 point)* If you walked outside right now and looked at license plates of cars passing by, you would not get observed probabilities anything close to most of the values above. For example, you probably don't see many cars with license plates ending with four letters. Why are the probabilities calculated above so different from those you would expect to observe in daily life?

6. *(5 points)* Prove that for any events $E_1, \ldots, E_n$, $\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \mathrm{P}\left(\bigcap_{i=1}^{n} E_i^c\right)$.

7. Currently (as of 6/23 at 8:15 am) there are 23 men and 18 women enrolled in STAT/MATH 394. Suppose we randomly choose 11 students from our class to form a soccer team.

   (a) *(2 points)* How many different teams can we have?

   (b) *(4 points)* How many different teams can we have with 6 women and 5 men?

   (c) *(4 points)* Given that we randomly select a team, what is the probability that the women outnumber the men?

   (d) *(4 points)* If we know that Sara Beck and Lisa Farino are on the team, what is the probability that the women outnumber the men?

   Now assume that a team has 1 goalkeeper, 4 defenders, 3 midfielders and 3 forwards. In other words, if we switch a defender and a midfielder, that counts as a different team.

   (e) *(4 points)* How many different teams can we have?

   (f) *(4 points)* How many different teams can we have with 6 women and 5 men?

[1] The information here is all true! See `http://seattletimes.nwsource.com/html/localnews/2003633850_licenseplates24m.html`.

(g) *(4 points)* Given that we randomly select a team, what is the probability that the women outnumber the men?

(h) *(4 points)* If we know that Sara Beck and Lisa Farino are on the team, what is the probability that the women outnumber the men?

8. In Alaska for 2010, the Democrats are strategizing to win the governorship, a US Senate Seat, and the majority of seats in the state legislature. Using the work of a pollster, they estimate their chances as follows:

   - Assuming Sarah Palin does not run for re-election, the probability of winning the governorship is 0.35.
   - The probability of winning the Senate seat is 0.35.
   - The probability of winning a majority in the state legislature is 0.5.
   - The probability of winning both the governorship and the Senate seat is 0.16.
   - The probability of winning a majority of the state legislature and the governorship is 0.15.
   - The probability of winning a majority in the state legislature and the Senate seat is 0.25.
   - The probability of winning only the governorship is 0.14.

   Questions:

   (a) *(3 points)* What is the probability that they will succeed in at most one of these endeavors?

   (b) *(3 points)* What is the probability that they will fail in at least one?

   (c) *(3 points)* What is the probability that they will succeed in exactly two of the three?

9. An urn contains a penny, a nickel, a dime, a quarter, a half dollar, and a Susan B. Anthony dollar. Two coins are drawn from this urn without replacement.

   (a) *(2 points)* List the elements of a sample space for this experiment and determine the sample size.

   (b) *(4 points)* Find the probability that the total value of the two coins selected is at most 30 cents.

   (c) *(4 points)* Find the probability that the total value of the two coins is an integral multiple of ten cents.

   (d) *(4 points)* Find the probability that the total value of the two coins selected is more than 15 cents.

10. If the letters in the word ADDYVAZEISAMATHSTUD are randomly rearranged, what is the probability of each of the following?:

    (a) *(5 points)* The vowels are all together at the beginning? (For this problem, assume that Y is a consonant)

    (b) *(5 points)* The vowels are all together?

11. **Extra Credit:** If five married (straight) couples are randomly seated in a row at the theater, what is the probability of each of the following?:

    (a) *(5 points)* The men and women alternate?

    (b) *(5 points)* No two men sit beside each other?

    (5 points) What are the answers to (a) and (b) above if we have n couples?

12. **Extra Credit:** *(5 points)* In Example 1.1.6 on page 2 of the notes, it states that the number of possible subsets of a set S is 2^n, where $n = |S|$ = the size of the set S. Prove this.

13. **Extra Credit:** In Example B.3.2 on page 235 of the notes, we see that there are $n!$ different ways to place n identical rooks onto an $n \times n$ chessboard. How many ways are there to do this if the rooks are *not* identical, but rather

    (a) *(2 points)* The n rooks are all different colors?

    (b) *(3 points)* There are k black and $n - k$ white rooks (where $k < n$?