STAT/MATH 394 A **Final Exam**
Derby **July 22, 2009 (Fake π Day)**

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Name

1. Suppose the joint pdf of X and Y is

$$f_{X,Y}(x, y) = 2, \qquad 0 < x < y, \quad 0 < y < 1.$$

   (a) *(2 points)* Are X and Y independent? Why or why not?

   (b) *(8 points)* Find the marginal pdfs of X and Y.

$$f_{X,Y}(x, y) = 2, \qquad 0 < x < y, \quad 0 < y < 1.$$

(c) *(4 points)* Find $F_X(\frac{1}{4})$.

(d) *(6 points)* Find $F_{X,Y}(\frac{1}{4}, \frac{3}{4})$.

2. Nate lectures on 394 material for 2 hours every Monday, Wednesday and Friday for six years. Students love these lectures so much that they rarely drop out, at a rate of 1 every two weeks following a Poisson process. Assume that enrollment is sufficiently large to model this as a Poisson process.

   (a) *(4 points)* What is the probability that at least 1 student leaves within 7 days?

   (b) *(4 points)* Given that at least 1 student left in a given week, what is the probability that at least 2 students left during that week?

   (c) *(4 points)* Sara Beck has had it with Nate's lectures and drops out. After she does so, what is the probability that the next student to leave will do so in less than 2 weeks?

(d) *(4 points)* A week goes by after Sara's departure without a student dropping the class. Given that this week has gone by with no student departures, what is the probability that there will be 2 or more additional weeks before the next student leaves the class? That is, given that there were no departures between time t and $t+1$ week, what is the probability that there are no drops between time $t+1$ week and $t+3$ weeks?

(e) *(4 points)* What is the probability that the next student to drop out will do so between 1 and 2 weeks from the time Sara drops the course?

3. In Nate's new house, the water heater is actually quite old. Its remaining lifetime X follows the pdf $f_X(x) = 4e^{-4x}$ for $x > 0$ in years.

   (a) *(2 points)* What is the name and parameter value of this distribution?

   (b) *(4 points)* What is the probability that his water heater lasts 6 months?

   (c) *(4 points)* Given that his water heater lasts for 6 months, what is the probability that it lasts for 4 more months after that?

(d) *(4 points)* What is the cumulative distribution function of X?

(e) *(4 points)* Actually, there are 8 appliances in Nate's house whose remaining lifetimes are independently and identically distributed according to this pdf $f_X(\cdot)$. What is the probability that 2 or more of them will last for another 6 months?

(f) *(2 points)* What is the name and parameter value of the distribution used to answer part (e) above? Note that this is not the same question as in part (a).

4. *(5 points)* Nate's wrench is missing somewhere in his new house, and it is equally likely to be in three possible rooms. Let $1 - \beta_i$ denote the probability that the wrench will be found upon a search of the i^{th} room when the wrench is, in fact, in that room, for $i \in \{1, 2, 3\}$. What is the conditional probability that the wrench is in the i^{th} room, given that a search of room 1 is unsuccessful, for all three values of i?

   NOTE: The constants β_i are called *overlook probabilities*, as they represent the probability of overlooking the wrench, which is attributable to the messiness of the room in question.

5. (a) *(5 points)* A woman challenges Michael Jackson's estate by claiming that he is the father of her child. An expert witness testifies that the length (in days) of pregnancy (i.e., time from impregnation to the delivery of the child) is approximately normally distributed with parameters $\mu = 270$ and $\sigma^2 = 100$. Evidence shows that Michael Jackson was out of the country (and away from this woman) during a period that began 289 days before the birth of the child and ended 236 days before the birth. If we assume that Jackson is, in fact, the father of the child, what is the probability that this woman could have had the very long or very short pregnancy indicated by the testimony?

   NOTE: For this problem, ignore the effect of modeling a discrete distribution with a continuous one (i.e., the continuity correction factor).

   (b) *(0 points)* Given this result, do you believe that Michael Jackson is the father of this woman's baby? Why or why not?

6. Let $X_1 \sim \mathrm{U}(2,3)$ be independent of $X_2 \sim \mathrm{U}(1,4)$.

   (a) *(8 points)* Determine $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$.

   (b) *(4 points)* Determine $f_{X_1,X_2}(x_1,x_2)$.

   (c) *(4 points)* Determine $F_{X_1,X_2}(x_1,x_2)$ for $x_1 \in (2,3)$ and $x_2 \in (1,4)$.

   (d) *(4 points)* Determine $F_{X_1,X_2}(x_1,x_2)$ for $x_1 \in (2,3)$ and $x_2 > 4$.

7. **Extra Credit:**

(a) *(2 points)* Let $X, Y \stackrel{iid}{\sim}$ Bernoulli(p). What is the distribution (name and parameter(s)) of $X + Y$? (No proof is needed)

(b) *(2 points)* Let $X \sim \mathrm{N}(5, \sigma^2 = 3)$. What is the distribution (name and parameter(s)) of $\left(\frac{X-5}{\sqrt{3}}\right)^2$? (No proof is needed)

(c) *(6 points)* Independent trials that result in a success with probability p are successively performed until a total of r successes is obtained. Give an argument that shows that the probability that exactly n trials are required is

$$\binom{n-1}{r-1} p^r (1-p)^{n-r}.$$